Analytic assessment of Laplacian estimates via novel variable inter-ring distances concentric ring electrodes

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Abstract— **Noninvasive concentric ring electrodes are a promising alternative to conventional disc electrodes. Currently, superiority of tripolar concentric ring electrodes over disc electrodes, in particular, in accuracy of Laplacian estimation has been demonstrated in a range of applications. In our recent work we have shown that accuracy of Laplacian estimation can be improved with multipolar concentric ring electrodes using a general approach to estimation of the Laplacian for an $(n + 1)$-polar electrode with n rings using the $(4n + 1)$-point method for $n \geq 2$. This paper takes the next step toward further improving the Laplacian estimate by proposing novel variable inter-ring distances concentric ring electrodes. Derived using a modified $(4n + 1)$-point method, linearly increasing inter-ring distances tripolar $(n = 2)$ and quadripolar $(n = 3)$ electrode configurations are analytically compared to their constant inter-ring distances counterparts using coefficients of the Taylor series truncation terms. Obtained results suggest that increasing inter-ring distances electrode configurations may decrease the truncation error of the Laplacian estimation resulting in more accurate Laplacian estimates compared to respective constant inter-ring distances configurations. For currently used tripolar electrode configuration the truncation error may be decreased more than two-fold while for the quadripolar more than seven-fold decrease is expected.**

I. Introduction

Electroencephalography (EEG) is an essential tool for brain and behavioral research as well as one of the mainstays of hospital diagnostic procedures and pre-surgical planning. Despite scalp EEG's many advantages end users struggle with its poor spatial resolution, selectivity and low signal-to-noise ratio that are critically limiting the research discovery and diagnosis [1]–[3]. In particular, EEG's poor spatial resolution is primarily due to (1) the blurring effects of the volume conductor with disc electrodes; and (2) EEG signals having reference electrode problems as idealized references are not available with EEG and interference on the reference electrode contaminates all other electrode signals [2]. The application of the surface Laplacian (the second spatial derivative of the potentials on the scalp surface) to EEG has been shown to alleviate the blurring effects enhancing the spatial resolution and selectivity, and reduce the reference problem [4]–[6].

Noninvasive concentric ring electrodes (CREs) can resolve the reference electrode problems since they act like closely spaced bipolar recordings [2]. Moreover, CREs are symmetrical alleviating electrode orientation problems [7]. They also act as spatial filters reducing the low spatial frequencies and increasing the spatial selectivity [7], [8]. Most importantly, tripolar CREs (TCREs; Fig. 1B) have been shown to estimate the surface Laplacian directly through the nine-point method, an extension of the five-point method (FPM) used for bipolar CREs, and significantly better than other electrode systems including bipolar and quasi-bipolar CRE configurations [9], [10]. Compared to EEG with conventional disc electrodes (Fig. 1A) Laplacian EEG via TCREs have been shown to have significantly better spatial selectivity (approximately 2.5 times higher), signal-to-noise ratio (approximately 3.7 times higher), and mutual information (approximately 12 times lower) [11]. Because of such unique capabilities TCREs have found numerous applications in a wide range of areas including brain–computer interface [12], [13], seizure onset detection [14], [15], detection of high-frequency oscillations and seizure onset zones [16], etc. These EEG applications of TCREs suggest the potential of CRE technology as well as the need for further improvement of CRE design.

In [17] we have shown that accuracy of Laplacian estimation can be improved with multipolar CREs. General approach to estimation of the Laplacian for an $(n + 1)$-polar electrode with n rings using the $(4n + 1)$-point method for $n \geq 2$ has been proposed. This approach allows cancellation of all the Taylor series truncation terms up to the order of $2n$ which has been shown to be the highest order achievable for a CRE with n rings [17]. Proposed approach was validated using finite element method (FEM) modeling. Multipolar concentric ring electrode configurations with n ranging from 1 ring (bipolar electrode configuration) to 6 rings (septapolar electrode configuration) were compared and obtained results suggested statistical significance of the increase in Laplacian accuracy caused by increase in the number of rings n [17].

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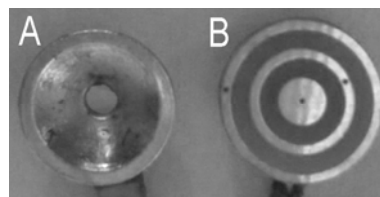
Figure 1. Conventional disc electrode (A) and tripolar concentric ring electrode (B).

To the best of the authors' knowledge, all the previous research on CREs was based on the assumption of constant inter-ring distances (distances between consecutive rings). This means that distances between the rings were not considered as a means of improving the accuracy of Laplacian estimation. This paper takes the next fundamental step toward further improving the Laplacian estimation accuracy by proposing novel variable inter-ring distances concentric ring electrodes. Laplacian estimates for linearly increasing inter-ring distances TCRE ($n = 2$) and quadripolar CRE (QCRE; $n = 3$) configurations are derived using a modified $(4n + 1)$-point method from [17] and directly compared to their constant inter-ring distances counterparts using coefficients of truncation terms from the Taylor series used in $(4n + 1)$-point method. Obtained results suggest that increasing inter-ring distances CRE configurations may offer more accurate Laplacian estimates compared to respective constant inter-ring distances CRE configurations.

II. MATERIALS AND METHODS

A. Notations and Preliminaries

In [17] general $(4n + 1)$-point method for constant inter-ring distances $(n + 1)$-polar CRE with n rings was proposed. It was derived using a regular plane square grid with all inter-point distances equal to r presented in Fig. 2.

It has been shown that for a case of multipolar CRE with n rings ($n \geq 2$) we obtain a set of n FPM equations for Laplacian potential Δv_0, one for each ring with radii ranging from r (v_0, $v_{r,1}$, $v_{r,2}$, $v_{r,3}$, and $v_{r,4}$ on Fig. 2) to nr (v_0, $v_{nr,1}$, $v_{nr,2}$, $v_{nr,3}$, and $v_{nr,4}$ on Fig. 2) around the point with potential v_0 [17]:
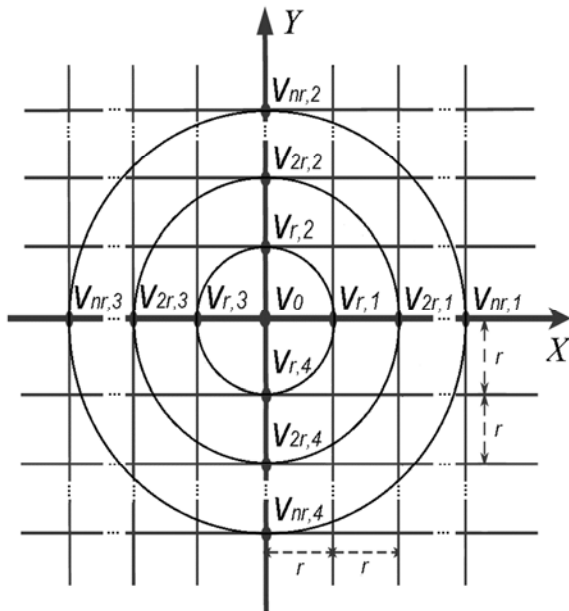
$$\frac{1}{2\pi}\int_0^{2\pi} v(nr,\theta)d\theta - v_0 = \frac{(nr)^2}{4}\Delta v_0$$
$$+\frac{(nr)^4}{4!}\int_0^{2\pi}\sum_{j=0}^{4}\sin^{4-j}(\theta)\cos^j(\theta)d\theta(\frac{\partial^4 v}{\partial x^{4-j}\partial y^j})$$
$$+\frac{(nr)^6}{6!}\int_0^{2\pi}\sum_{j=0}^{6}\sin^{6-j}(\theta)\cos^j(\theta)d\theta(\frac{\partial^6 v}{\partial x^{6-j}\partial y^j}) + ... \quad (1)$$

To estimate the Laplacian for this general case the n equations are combined in a way that cancels all the truncation terms up to the highest order that can be achieved for n rings (equal to $2n$ as shown in [17]) increasing the accuracy of the Laplacian estimate. In order to find such a combination we arrange the coefficients l^k of the truncation terms $\frac{(lr)^k}{k!}\int_0^{2\pi}\sum_{j=0}^{k}\sin^{k-j}(\theta)\cos^j(\theta)d\theta(\frac{\partial^k v}{\partial x^{k-j}\partial y^j})$ for order k ranging in increments of 2 from 4 to $2n$ and ring radius multiplier l ranging from 1 to n into an $n - 1$ by n matrix A that is a function only of the number of the rings n:

$$A = \begin{pmatrix} 1 & 2^4 & \cdots & n^4 \\ 1 & 2^6 & \cdots & n^6 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{2n} & \cdots & n^{2n} \end{pmatrix} \quad (2)$$

A matrix equation of the form $A\bar{x} = \bar{0}$ is equivalent to a homogeneous system of linear equations where $\bar{0}$ is the $(n - 1)$-dimensional zero vector and $\bar{x}$ is the n-dimensional vector that allows the cancellation of all the truncation terms up to the order of $2n$ by setting the linear combination of n coefficients l^k corresponding to all ring radii for each order k equal to 0 [17].

B. Variable (Linearly Increasing) Inter-ring Distances CRE

We consider the case of CRE configurations with variable inter-ring distances that increase or decrease linearly the further the concentric ring lies from the central disc. To modify the $(4n + 1)$-point method from [17] to the case of linearly increasing inter-ring distances, the distance between the central point with potential v_0 and four points on the smallest concentric ring is set equal to r. The distance between the first and the second smallest concentric ring is set equal to $2r$, etc. In this case the sum of all the inter-ring distances to the largest, n-th, outer ring can be obtained using the formula for the n-th term of the triangular number sequence that describes the sum of all points in a triangular grid where the first row contains a single point and each subsequent row contains one more point than the previous one to be equal to $n(n + 1)/2$ [18]. Therefore, modified matrix A of truncation term coefficients l^k from (2) for linearly increasing inter-ring distances CRE is equal to:

Figure 2. Regular plane square grid with inter-point distances equal to r.

$$A' = \begin{pmatrix} 1 & 3^4 & \cdots & (\frac{n(n+1)}{2})^4 \\ 1 & 3^6 & \cdots & (\frac{n(n+1)}{2})^6 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 3^{2n} & \cdots & (\frac{n(n+1)}{2})^{2n} \end{pmatrix} \quad (3)$$

C. Analysis of the Taylor Series Truncation Term Coefficients

Variable inter-ring distances CREs have the same number of rings and, therefore, the same number and order of truncation terms in Laplacian estimates as their constant inter-ring distances counterparts. Therefore, constant and variable inter-ring distances CRE configurations can be directly compared by assessing the coefficients at the respective truncation terms that comprise the truncation error of the Laplacian estimation.

Analyzing those coefficients will allow us to determine which electrode configuration allows minimizing the truncation error resulting in more accurate Laplacian estimate. Performing this kind of analysis for increasing and constant inter-ring distances TCREs and QCREs would allow assessing the potential of variable inter-ring distances CREs.

First, we derive the coefficients of the truncation terms for TCRE and QCRE configurations with increasing and constant inter-ring distances as functions of the order of the truncation term, k, under the following conditions: the largest, outer ring radius equals to $6r$ and relative locations of concentric rings, with respect to the central disc, are as shown in Fig. 3. Same dimensions of all the different CRE configurations considered ensured direct comparability of results.

For constant inter-ring distances TCREs and QCREs the coefficients used to combine the differences between the concentric ring potentials and the central disc potential into a Laplacian estimate can be derived using the approach proposed in [17] by finding the null space of matrix A from (2) for $n = 2$ and $n = 3$ respectively: $(16, -1)$ for TCRE and $(270, -27, 2)$ for QCRE configurations. Derivation of Laplacian estimate coefficients for increasing inter-ring distances TCREs and QCREs configurations was performed using the approach proposed in this paper by finding the null space of matrix A' from (3) for $n = 2$ and $n = 3$ respectively: $(81, -1)$ for TCRE and $(4374, -70, 1)$ for QCRE configurations. This approach cancels all the truncation terms up to the order of $2n$ which has been shown to be the highest order achievable for a CRE with n rings [17]. In the case of TCREs ($n = 2$) this corresponds to cancellation of the fourth order leaving truncation terms of orders 6 and higher. Assuming that our TCRE has two rings with radii αr and βr respectively such that $\beta > \alpha$, for each ring we take the integral along the circle with the corresponding radius of the Taylor series in a manner identical to deriving (1) to obtain:
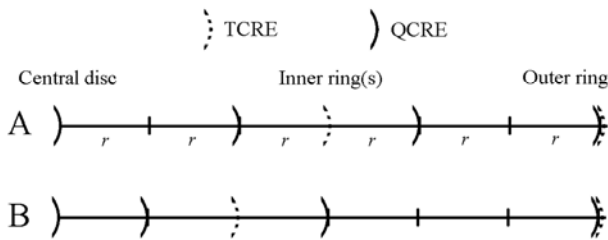

Figure 3.   Relative locations of rings for constant (panel A) and increasing (panel B) inter-ring distances TCRE and QCRE configurations.

$$\frac{1}{2\pi}\int_0^{2\pi}v(\alpha r,\theta)d\theta = v_0 + \frac{(\alpha r)^2}{4}\Delta v_0$$
$$+ \frac{(\alpha r)^4}{4!}\int_0^{2\pi}\sum_{j=0}^4\sin^{4-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^4 v}{\partial x^{4-j}\partial y^j}\right)$$
$$+ \frac{(\alpha r)^6}{6!}\int_0^{2\pi}\sum_{j=0}^6\sin^{6-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^6 v}{\partial x^{6-j}\partial y^j}\right) + ... \quad (4)$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}v(\beta r,\theta)d\theta = v_0 + \frac{(\beta r)^2}{4}\Delta v_0$$
$$+ \frac{(\beta r)^4}{4!}\int_0^{2\pi}\sum_{j=0}^4\sin^{4-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^4 v}{\partial x^{4-j}\partial y^j}\right)$$
$$+ \frac{(\beta r)^6}{6!}\int_0^{2\pi}\sum_{j=0}^6\sin^{6-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^6 v}{\partial x^{6-j}\partial y^j}\right) + ... \quad (5)$$

For constant inter-ring distances TCREs we combine equations (4) and (5) by multiplying (4) by 16, multiplying (15) by −1, and adding the two resulting products together solving the sum for the Laplacian Δv_0:

$$\Delta v_0 = \frac{1}{\left(\frac{16\alpha^2 - \beta^2}{4}\right)r^2}[16(v_{MR} - v_0) - (v_{OR} - v_0)$$
$$+ \frac{(16\alpha^4 - \beta^4)r^4}{4!}\int_0^{2\pi}\sum_{j=0}^4\sin^{4-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^4 v}{\partial x^{4-j}\partial y^j}\right)$$
$$+ \frac{(16\alpha^6 - \beta^6)r^6}{6!}\int_0^{2\pi}\sum_{j=0}^6\sin^{6-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^6 v}{\partial x^{6-j}\partial y^j}\right) + ...] \quad (6)$$

where $v_{MR} = \frac{1}{2\pi}\int_0^{2\pi}v(\alpha r,\theta)d\theta$ is the potential on the middle ring of the radius αr and $v_{OR} = \frac{1}{2\pi}\int_0^{2\pi}v(\beta r,\theta)d\theta$ is the potential on the outer ring of the radius βr.

For increasing inter-ring distances TCREs equations (4) and (5) have to be combined with the coefficients 81 and −1 respectively resulting in:

$$\Delta v_0 = \frac{1}{\left(\frac{81\alpha^2 - \beta^2}{4}\right)r^2}[81(v_{MR} - v_0) - (v_{OR} - v_0)$$
$$+ \frac{(81\alpha^4 - \beta^4)r^4}{4!}\int_0^{2\pi}\sum_{j=0}^4\sin^{4-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^4 v}{\partial x^{4-j}\partial y^j}\right)$$
$$+ \frac{(81\alpha^6 - \beta^6)r^6}{6!}\int_0^{2\pi}\sum_{j=0}^6\sin^{6-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^6 v}{\partial x^{6-j}\partial y^j}\right) + ...] \quad (7)$$

Now we can express the coefficients $c(k)$ of truncation terms with the general form $\frac{c(k)r^{k-2}}{k!}\int_0^{2\pi}\sum_{j=0}^k\sin^{k-j}(\theta)\cos^j(\theta)d\theta\left(\frac{\partial^k v}{\partial x^{k-j}\partial y^j}\right)$ as the function of the truncation term order k. For constant inter-ring distances TCRE and α and β equal to 3 and 6 respectively

(Fig. 3A) we obtain $c_C^{TCRE}(k) = \dfrac{4\left(16 \cdot 3^k - 6^k\right)}{16 \cdot 3^2 - 6^2} = \dfrac{16 \cdot 3^k - 6^k}{27}$ for even $k \geq 6$. For increasing inter-ring distances TCRE and α and β equal to 2 and 6 respectively (Fig. 3B) we obtain

$$c_I^{TCRE}(k) = \frac{4\left(81 \cdot 2^k - 6^k\right)}{81 \cdot 2^2 - 6^2} = \frac{81 \cdot 2^k - 6^k}{72} \text{ for even } k \geq 6.$$

The same steps can be taken to derive the truncation term coefficient functions for increasing and constant inter-ring distances QCREs ($n = 3$) cancelling the truncation terms up to the sixth order. For constant inter-ring distances QCRE coefficients (270, –27, 2) are used to combine potentials on three rings with radii $2r$, $4r$, and $6r$ (Fig. 3A) and the central disc resulting in the following for even $k \geq 8$

$$c_C^{QCRE}(k) = \frac{4\left(270 \cdot 2^k - 27 \cdot 4^k + 2 \cdot 6^k\right)}{270 \cdot 2^2 - 27 \cdot 4^2 + 2 \cdot 6^2} = \frac{270 \cdot 2^k - 27 \cdot 4^k + 2 \cdot 6^k}{180}.$$

For increasing inter-ring distances QCRE coefficients (4374, –70, 1) are used to combine potentials on three rings with radii r, $3r$, and $6r$ (Fig. 3B) and the central disc resulting in the following for even $k \geq 8$:

$$c_I^{QCRE}(k) = \frac{4\left(4374 \cdot 1^k - 70 \cdot 3^k + 1 \cdot 6^k\right)}{4374 \cdot 1^2 - 70 \cdot 3^2 + 1 \cdot 6^2} = \frac{4374 - 70 \cdot 3^k + 6^k}{945}.$$

The ratio of truncation term coefficient functions for constant inter-ring distances to increasing inter-ring distances TCRE configurations is the following for even $k \geq 6$:

$$r_{CI}^{TCRE}(k) = c_C^{TCRE}(k) \div c_I^{TCRE}(k) = \frac{8\left(16 \cdot 3^k - 6^k\right)}{3\left(81 \cdot 2^k - 6^k\right)} \tag{8}$$

In a similar way, the ratio of truncation term coefficient functions for constant inter-ring distances to increasing inter-ring distances QCRE configurations is the following for even $k \geq 8$:

$$r_{CI}^{QCRE}(k) = c_C^{QCRE}(k) \div c_I^{QCRE}(k) = \frac{21\left(270 \cdot 2^k - 27 \cdot 4^k + 2 \cdot 6^k\right)}{4\left(4374 - 70 \cdot 3^k + 6^k\right)} \tag{9}$$

III. RESULTS

Plots of both functions from (8) and (9) are presented in Fig. 4 for even truncation term order k ranging from 6 to 30 and from 8 to 30 respectively.

While the signs of the truncation term coefficients are consistent for both constant and increasing inter-ring distances CRE configurations (all negative for TCREs and all positive for QCREs), Fig. 4 serves a three-fold purpose. First, it shows that absolute values of coefficients are larger for constant inter-ring distances CRE configurations since ratios of truncation term coefficients for constant inter-ring distances CRE configurations over corresponding increasing inter-ring distances CRE configurations are all larger than 1. Therefore, truncation errors for constant inter-ring distances CRE configurations are greater than the ones for corresponding increasing inter-ring distances CRE configurations which results in more accurate Laplacian estimates for increasing inter-ring distances CRE configurations. Second, Fig. 4 shows that the ratios of truncation term coefficients are higher for QCREs than for TCREs.
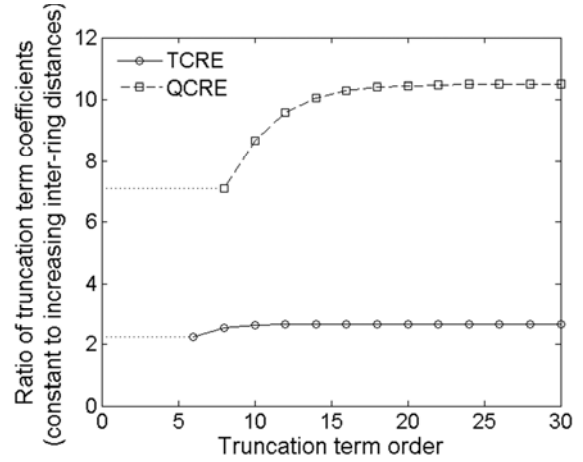

Figure 4. Ratios of constant to increasing inter-ring distances truncation term coefficients as functions of the truncation term order.

Therefore, the improvement in Laplacian accuracy is likely to become more significant with the increase in the number of rings. Third, Fig. 4 shows that all the coefficient ratios increase with the increase of the truncation term order but according to [19] for Taylor series "higher-order terms usually contribute negligibly to the final sum and can be justifiably discarded." Therefore, we will consider the coefficient ratios for the lowest nonzero truncation term for TCRE (sixth order) and QCRE (eighth order) configurations equal to 2.25 and 7.11 respectively (dotted lines in Fig. 4) as the ones that contribute the most to the truncation error. If we take weighted arithmetic means of all the truncation term coefficient ratios from Fig. 4 for truncation term orders up to 30 with weights derived from an exponential decay model with unit original amount and decay rate equal to –1 to account for decreasing contribution of higher order terms we obtain weighted average ratios of 2.37 and 7.83 respectively.

IV. DISCUSSION

The contribution of this paper is twofold. First, novel variable inter-ring distances CREs are proposed as opposed to all the previous research on CREs that, to the best of the authors' knowledge, was based on the assumption of constant inter-ring distances. Laplacian estimates for variable inter-ring distances CREs are derived using a modified $(4n + 1)$-point method from [17] for any given number of rings n. Second, accuracy of Laplacian estimates corresponding to constant and linearly increasing inter-ring distances TCREs and QCREs is assessed analytically with obtained results suggesting that for TCREs the truncation error may be decreased more than two-fold (2.25) while for QCREs more than seven-fold (7.11) decrease is expected.

Analytical results obtained in this study are based on our hypothesis that the ratios of constant inter-ring distances truncation term coefficient functions over the increasing inter-ring distances truncation term coefficient functions for TCRE and QCRE configuration will be comparable to the respective ratios of Laplacian estimation error obtained using FEM modelling or prototypes of increasing inter-ring distances CREs on real life data, both phantom and from human subjects. This hypothesis stems from the fact that for Laplacian estimates obtained using the $(4n + 1)$-point method

truncation terms constitute the primary component of the estimation error.

The type of truncation term coefficient analysis that was used in this study would not have been feasible in our previous works. For example, in [17], where multipolar CRE configurations ranging from bipolar ($n = 1$) to septapolar ($n = 6$) were compared using FEM modelling, Laplacian estimates for different CRE configurations had different numbers of truncation terms (one truncation term less for each additional concentric ring causing an increase in Laplacian estimation accuracy) which made analytical comparison of truncation term coefficients for different CRE configurations infeasible. In this study proposed variable inter-ring distances CREs have the same numbers of rings and, therefore, the same numbers (and orders) of truncation terms in respective Laplacian estimates as their constant inter-ring distances counterparts which allowed us to quantify the expected improvement in Laplacian estimation accuracy analytically.

Further investigation is needed to confirm the obtained results. The plan for future work includes several directions and is based on limitations of the current study. The main limitation of the proposed ($4n + 1$)-point method is that the width of the concentric rings and the diameter of the central disc are not taken into account and therefore cannot be optimized. To pursue our ultimate goal of being able to determine sub optimal CRE designs for specific applications these two parameters need to be included into future modifications of the ($4n + 1$)-point method and corresponding FEM models along with the currently included number of rings, size of the electrode, and, as proposed in this study, inter-ring distances. Another limitation is that while this study proposes novel variable inter-ring distances CREs, only linearly increasing distances are considered. The solution to the general inter-ring distances optimization problem is likely to result in nonlinear relationship which is why solving this general problem is the second direction of the future work. Third direction is to create prototypes of variable inter-ring distances CREs with 2 and more rings and test them on real life data, both phantom and from human subjects. This direction is critical since obtained results suggest that variable inter-ring distances CREs may result in more accurate Laplacian estimates. This raises the question of how small can the distances between concentric rings get before partial shorting due to salt bridges becomes significant enough to affect Laplacian estimation. Moreover, these prototypes would allow investigating the translation of ratios of truncation term coefficients assessed in this study into improvement of spatial selectivity, signal-to-noise ratio, source mutual information, etc. in a similar was to how it was investigated for Laplacian EEG via TCREs compared to EEG with conventional disc electrodes [11].

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